NONEXISTENCE OF LINEAR OPERATORS EXTENDING LIPSCHITZ (PSEUDO)METRICS

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Abstract. We provide an example of a zero-dimensional compact metric space X and its closed subspace A such that there is no continuous linear extension operator for the Lipschitz pseudometrics on A to the Lipschitz pseudometrics on X. The construction is based on results of A. Brudnyi and Yu. Brudnyi concerning linear extension operators for Lipschitz functions.

1. Introduction

The problem of extensions of metrics has a long history. It was F. Hausdorff who first proved that any continuous metric defined on a closed subset of a metrizable space can be extended to a continuous metric defined on the whole space. C. Bessaga first considered the problem of existence of linear extension operators for metrics (see [1] and [2]) and provided a partial solution of this problem. The problem was completely solved by T. Banakh [3] (see also a short proof in [4]).

The problem of extension of Lipschitz and uniform (pseudo)metrics has been considered in [5]. It is known that any Lipschitz pseudometric defined on a closed subset of a metric space admits an extension which is a Lipschitz pseudometric defined on the whole space. In this note we consider a problem of existence of linear extension operators for Lipschitz pseudometrics. Up to the author's knowledge, no affirmative results are obtained in this direction. Our aim here is to construct a counterexample: there exists a subset of a zero-dimensional compact metric space for which there is no such an extension operator. The example is based on the results from [6] concerning the linear extension operators for Lipschitz functions.

Note that conditions for existence of extensions of continuous functions are often equivalent to those for existence of extensions of continuous pseudometrics (see, e.g., [7], [8], [9], [10]). It turns out that, in the case of linear extension operators, one is able to proceed at least in one direction, from pseudometrics to functions.

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2. Preliminaries

Let (X,d) be a compact metric space. Given a subset A of X, we say that a pseudometric ϱ on A is said to be *Lipschitz* if there is $C>0$ such that $d(x,y)\le C\varrho(x,y)$, for any $x,y\in A$. Also, a function $f\colon A\to\mathbb{R}$ is *Lipschitz* if if there is $C>0$ such that $|f(x)-f(y)|\le Cd(x,y)$, for every $x,y\in A$. Denote by $\mathcal{PM}_{\mathrm{Lip}}(A)$ (respectively $\mathcal{M}_{\mathrm{Lip}}(A)$, $\mathcal{C}_{\mathrm{Lip}}(A)$) the set of all Lipschitz pseudometrics (respectively metrics, functions) on A. The set $\mathcal{PM}_{\mathrm{Lip}}(A)$ (respectively $\mathcal{C}_{\mathrm{Lip}}(A)$) is a cone (respectively linear space) with respect to the operations of pointwise addition and multiplication by scalar. We endow $\mathcal{PM}_{\mathrm{Lip}}(A)$ with the norm $\|\cdot\|_{\mathcal{PM}_{\mathrm{Lip}}(A)}$,

$$\|d\|_{\mathcal{PM}_{\mathrm{Lip}}(A)}=\sup\left\{\frac{d(x,y)}{\varrho(x,y)}\mid x\neq y\right\}$$

and $\mathcal{C}_{\mathrm{Lip}}(A)$ with the seminorm $\|\cdot\|_{\mathcal{C}_{\mathrm{Lip}}(A)}$,

$$\|f\|_{\mathcal{C}_{\mathrm{Lip}}(A)}=\sup\left\{\frac{|f(x)-f(y)|}{\varrho(x,y)}\mid x\neq y\right\}$$

(in the sequel, we abbreviate $\|\cdot\|_{\mathcal{PM}_{\mathrm{Lip}}(A)}$ and $\|\cdot\|_{\mathcal{C}_{\mathrm{Lip}}(A)}$ to $\|\cdot\|_A$).

We say that a map $u\colon\mathcal{PM}_{\mathrm{Lip}}(A)\to\mathcal{PM}_{\mathrm{Lip}}(X)$ is an *extension operator* for Lipschitz pseudometrics if the following holds:

(1) u is linear (i.e. $u(d_1+d_2)=u(d_1)+u(d_2)$, $u(\lambda d)=\lambda u(d)$ for every $d_1,d_2\in\mathcal{PM}_{\mathrm{Lip}}(A)$, $\lambda\in\mathbb{R}_+$);
(2) $u(d)|(A\times A)=d$, for every $d\in\mathcal{PM}_{\mathrm{Lip}}(A)$;
(3) u is continuous in the sense that $\|u\|=\sup\{\|u(d)\|_X\mid\|d\|_A\le 1\}$ is finite.

This definition is a natural counterpart of those introduced in [6] for the extensions of Lipschitz functions. The following notation is introduced in [6]:

$$\lambda(A,X)=\inf\{\|u\|\mid u\text{ is a linear extension operator from }\mathcal{C}_{\mathrm{Lip}}(A)\text{ to }\mathcal{C}_{\mathrm{Lip}}(X)\}.$$

Similarly, we put

$$\begin{aligned}\Lambda(A,X)=\inf\{\|u\|\mid u&\text{ is a linear extension operator from}\\ &\mathcal{PM}_{\mathrm{Lip}}(A)\text{ to }\mathcal{PM}_{\mathrm{Lip}}(X)\}.\end{aligned}$$

It can be easily proved (cf. [6]) that $\Lambda(X)=\sup\{\Lambda(A,X)\mid A\subset X\}$ is a bi-Lipschitz invariant of X.

3. Auxiliary results

Given a metric space $X=(X,d)$ and $c>0$, we denote by cX the space (X,cd).

Lemma 3.1. *For any metric space (X,d), subset $A\subset X$, and $c>0$, we have $\lambda(cS,cX)=\lambda(S,X)$.*

Proof. Let $\varphi \in \mathcal{C}_{\mathrm{Lip}}(cS)$ and $\|\varphi\|_{cS} = K$. Then φ can be also considered as an element of $\mathcal{C}_{\mathrm{Lip}}(S)$ with $\|\varphi\|_S = K/c$. There exists an extension $\bar{\varphi}\colon X \to \mathbb{R}$ of φ with $\|\bar{\varphi}\|_X \le (K\lambda(S,X))/c$. Considering $\bar{\varphi}$ as an element of $\mathcal{C}_{\mathrm{Lip}}(cX)$, we see that $\|\bar{\varphi}\|_{cX} \le (K\lambda(S,X))$. Therefore $\lambda(cS,cX) \le \lambda(S,X)$. Arguing similarly, one can prove te opposite inequality. $\square$

Lemma 3.2. *Let a metric pair (S_1, X_1) be a retract of a metric pair (S,X) under a 1-Lipschitz retraction. Then $\lambda(S_1,X_1) \le \lambda(S,X)$.*

Proof. Let $r\colon X \to X_1$ be a 1-Lipschitz retraction such that $r(S) = S_1$. Given a Lipschitz function $f\colon S_1 \to \mathbb{R}$, we see that $f \circ (r|S)$ is a Lipschitz function on S with $\|f \circ (r|S)\|_S = \|f\|_{S_1}$. There is an extension $g\colon X \to \mathbb{R}$ of $f \circ (r|S)$ with $\|g\|_X \le \lambda(S,X)\|f \circ (r|S)\|_S$. Then $g|X_1$ is an extension of f over X_1 with $\|g|X_1\|_{X_1} \le \lambda(S,X)\|f\|_{S_1}$. $\square$

Proposition 3.3. *Let S be a closed subset of a compact metric space X with $|S| \ge 2$. The following are equivalent:*

(1) *there exists a continuous linear extension operator from $\mathcal{PM}_{\mathrm{Lip}}(S)$ to $\mathcal{PM}_{\mathrm{Lip}}(X)$;*
(2) *there exists a continuous linear extension operator from $\mathcal{M}_{\mathrm{Lip}}(S)$ to $\mathcal{M}_{\mathrm{Lip}}(X)$.*

Proof. (1)⇒(2). Let $u\colon \mathcal{PM}_{\mathrm{Lip}}(S) \to \mathcal{PM}_{\mathrm{Lip}}(X)$ be a continuous linear extension operator.

Note first that there exists $\tilde{\varrho} \in \mathcal{PM}_{\mathrm{Lip}}(X)$ such that $\tilde{\varrho}(x,y) = 0$ if and only if $(x,y) \in (S \times S) \cup \Delta_X$ (by Δ_X we denote the diagonal of X). In order to construct $\tilde{\varrho}$, for any $x, y \in X \setminus S$ with $x \ne y$, consider the pseudometric ϱ_{xy} on $S \cup \{x,y\}$ defined by

$$\varrho_{xy}|(S \times S) = 0,\ \ \varrho_{xy}(x,y) = \varrho_{xy}(x,s) = \varrho_{xy}(y,s) = 0$$

for any $s \in S$. Denoting by d the original metric on X we see that

$$\begin{aligned} \varrho_{xy}(x,y) &= 1 = (1/d(x,y))d(x,y),\\ \varrho_{xy}(x,s) &= 1 \le (1/d(x,S))d(x,s)\\ \varrho_{xy}(y,s) &= 1 \le (1/d(y,S))d(y,s) \end{aligned}$$

for any $s \in S$. Therefore, ϱ_{xy} is a Lipschitz pseudometric with the Lipschitz constant $\max\{1/d(x,y), 1/d(x,S), 1/d(y,S)\}$. By the result of Luukkainen [5], there exists a Lipschitz pseudometric, $\tilde{\varrho}_{xy}$, on X which is an extension of ϱ_{xy}.

There exist (necessarily disjoint) neighborhoods, U_{xy} and V_{xy}, of x and y respectively such that $\tilde{\varrho}_{xy}(x',y') \ne 0$, for every $x' \in U_{xy}$ and $y' \in V_{xy}$. The family

$$\{U_{xy} \times V_{xy} \mid x,y \in ((X \setminus S) \times (X \setminus S)) \setminus \Delta_X\}$$

forms an open cover of $((X \setminus S) \times (X \setminus S)) \setminus \Delta_X$ and, by separability of the latter set, there exists a sequence (x_i, y_i) in $((X \setminus S) \times (X \setminus S)) \setminus \Delta_X$ such that

$$\bigcup \{U_{x_i y_i} \times V_{x_i y_i} \mid i \in \mathbb{N}\} = ((X \setminus S) \times (X \setminus S)) \setminus \Delta_X.$$

Let $\tilde{\varrho} = \displaystyle\sum_{i=1}^{\infty} \frac{\tilde{\varrho}_{x_i y_i}}{2^i \|\tilde{\varrho}_{x_i y_i}\|_X}$. Then, obviously, $\tilde{\varrho} \in \mathcal{PM}_{\mathrm{Lip}}(X)$ is as required.

Now define an operator $\tilde{u} \colon \mathcal{M}_{\mathrm{Lip}}(S) \to \mathcal{M}_{\mathrm{Lip}}(X)$ as follows. Let $x_0, y_0 \in S$, $s_0 \neq y_0$. Let $\tilde{u}(\delta) = u(\delta) + \delta(x_0, y_0)\tilde{\varrho}$, for any $\delta \in \mathcal{M}_{\mathrm{Lip}}(S)$. We leave to the reader an easy verification that $\tilde{u}$ is a continuous extension operator.

(2)⇒(1). We are going to show that $\mathcal{M}_{\mathrm{Lip}}(S)$ is dense in $\mathcal{PM}_{\mathrm{Lip}}(S)$. Let $\varepsilon > 0$. Given $\varrho \in \mathcal{PM}_{\mathrm{Lip}}(S)$, we see that $\varrho_1 = \varrho + \varepsilon d'$, where d' is the original metric on S, is an element of $\mathcal{M}_{\mathrm{Lip}}(S)$ with $\|\varrho - \varrho_1\|_S \leq \varepsilon$.

Let $u \colon \mathcal{M}_{\mathrm{Lip}}(S) \to \mathcal{M}_{\mathrm{Lip}}(X)$ be a continuous linear extension operator. Since $\mathcal{M}_{\mathrm{Lip}}(S)$ is dense in $\mathcal{PM}_{\mathrm{Lip}}(S)$ and the space $\mathcal{PM}_{\mathrm{Lip}}(X)$ is complete, there exists a unique continuous extension, $\tilde{u} \colon \mathcal{PM}_{\mathrm{Lip}}(S) \to \mathcal{PM}_{\mathrm{Lip}}(X)$, of u. Obviously, $\tilde{u}$ is a continuous linear extension operator.

□

4. Main result

Theorem 4.1. *There exists a closed subspace A of a zero-dimensional compact metric space X for which there is no extension operator for Lipschitz pseudometrics.*

Proof. We recall some results from [6]. Let $\mathbb{Z}^n_1(l)$ stand for $\mathbb{Z}^n \cap [-l, l]^n$ endowed with the ℓ_1-metric.

It is proved [6, Lemma 10.5] that, for any natural n, there exists $c_1 > 0$, $l(n) > 0$, and a subset $Y_n \subset \mathbb{Z}^n_1(l(n))$ such that

$$\lambda(Y_n, \mathbb{Z}^n_1(l(n))) \geq c_1\sqrt{n}. \tag{4.1}$$

Let

$$X = \left(\coprod_{n=1}^{\infty} \frac{1}{nl(n)} \mathbb{Z}^n_1(l(n))\right) / \sim,$$

where $\sim$ is the equivalence relation which identifies all the origins, be the bouquet of the spaces $\mathbb{Z}^n_1$, $n \in \mathbb{N}$. We naturally identify every $\frac{1}{nl(n)}\mathbb{Z}^n_1(l(n)))$, $n \in \mathbb{N}$, with its copy, X_n, in X. The space X is endowed with the maximal metric, ϱ, inducing the original metric on X_n, for every $n \in \mathbb{N}$. Obviously, X is a compact metric space. One can easily see that X is zero-dimensional.

By $\mathcal{C}_{\mathrm{Lip}}(X, 0)$ we denote the set of functions from $\mathcal{C}_{\mathrm{Lip}}(X)$ that vanish at $0 \in X$ and by $\mathcal{C}^+_{\mathrm{Lip}}(X, 0)$ we denote the set of nonnegative functions from $\mathcal{C}_{\mathrm{Lip}}(X, 0)$.

Let $S_n = \frac{1}{nl(n)} Y_n \subset X_n$, $n \in \mathbb{N}$. Suppose that there exists an extension operator $u \colon \mathcal{PM}_{\mathrm{Lip}}(S) \to \mathcal{PM}_{\mathrm{Lip}}(X)$, where $S = \coprod_{n=1}^{\infty} S_n \subset X$. Define a map

$v\colon \mathcal{C}_{\mathrm{Lip}}(S) \to \mathcal{C}_{\mathrm{Lip}}(X)$ by the following manner. First, let $f \in \mathcal{C}^+_{\mathrm{Lip}}(S,0)$, then the function

$$d_f\colon S \times S \to \mathbb{R}^+,\ d_f(x,y) = |f(x) - f(y)|,\ x,y \in X,$$

is a Lipschitz pseudometric on X and we let $v(f)(x) = u(d_f)(x,0)$. Given $g \in \mathcal{C}_{\mathrm{Lip}}(S,0)$, represent g as the difference, $g = g_1 - g_2$, where $g_1, g_2 \in \mathcal{C}^+_{\mathrm{Lip}}(S,0)$ (say, $g_1 = (|g|+g)/2$, $g_2 = (|g|-g)/2$), and let $v(g) = v(g_1) - v(g_2)$. Note that v is well-defined: if $g = g_1 - g_2 = g_1' - g_2'$, where $g_1, g_2, g_1', g_2' \in \mathcal{C}^+_{\mathrm{Lip}}(S,0)$, then $g_1 + g_2' = g_1' + g_2$, whence, by the linearity of u, we have

$$\begin{aligned} v(g_1)(x) + v(g_2')(x) =& u(d_{g_1})(x,0) + u(d_{g_2'})(x,0) = u(d_{g_1} + d_{g_2'})(x,0) \\ =& g_1(x) + g_2'(x) = g_1'(x) + g_2(x) = u(d_{g_1'} + d_{g_2})(x,0) \\ =& u(d_{g_1'})(x,0) + u(d_{g_2})(x,0) = v(g_1')(x) + v(g_2)(x), \end{aligned}$$

i.e., $v(g_1) + v(g_2') = v(g_1') + v(g_2)$.

If $h \in \mathcal{C}_{\mathrm{Lip}}(S)$, then $h - h(0) \in \mathcal{C}_{\mathrm{Lip}}(S,0)$ and we put $v(h) = v(h - h(0)) + h(0)$. By direct verification we show that $v\colon \mathcal{C}_{\mathrm{Lip}}(S) \to \mathcal{C}_{\mathrm{Lip}}(X)$ is a linear extension operator with $\|v\| < \infty$. Therefore, $\lambda(S,X) < \infty$.

For every n, denote by $r_n\colon X \to X_n$ the retraction that sends the complement to X_n to $0 \in X_n$. Then, evidently, r_n is a 1-Lipschitz retraction, $r_n(S) = S_n$ and, by Lemma 3.2, $\lambda(S_n, X_n) \le \lambda(S,X)$. This obviously contradicts to inequality (4.1). □

It follows from Proposition 3.3 that for the subset S of the space X from the proof of Theorem 4.1 there is no continuous linear operator extending Lipschitz metrics.

5. Remarks and open problems

It easily follows from the proof of Theorem 4.1 that $\lambda(S,X) < \infty$, whenever $\Lambda(S,X) < \infty$, for any subset S of a metric space X.

Question 5.1. Compare $\Lambda(S,X)$ and $\lambda(S,X)$.

We conjecture that $\Lambda(S,X) = \lambda(S,X)$; if, however, this is not the case, one can ask for a pseudometric counterpart of any result on linear extensions of Lipschitz functions. As an example, we formulate the following question inspired by results from [11].

Question 5.2. Let (X,d) be a metric space and $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a concave non-decreasing function with $\omega(0) = 0$. The function $d_\omega = \omega \circ d$ is a metric on X. Are the properties $\Lambda(X, d_\omega) < \infty$ and $\Lambda(X,d) < \infty$ equivalent?

It is proved in [12] that there exists a linear operator which extends partial pseudometrics with variable domain. The following question is, in some sense, a

strenthening of Question 5.1. Given a compact metric space X, we let

$$\mathcal{PM}_{\mathrm{Lip}} = \bigcup\{\mathcal{PM}_{\mathrm{Lip}}(A) \mid A \text{ is a nonempty closed subset of } X\}.$$

One can endow $\mathcal{PM}_{\mathrm{Lip}}$ with the following metric, D:

$$D(\varrho_1, \varrho_2) = \inf\{\|\tilde{\varrho}_1 - \tilde{\varrho}_2\|_X \mid \tilde{\varrho}_i \text{ is a Lipschitz extension of } \varrho_i\}.$$

Question 5.3. Suppose that, for a metric space X, $\Lambda(X) < \infty$. Is there a continuous linear extension operator for partial Lipschitz pseudometrics on X, i.e., a map $u\colon \mathcal{PM}_{\mathrm{Lip}} \to \mathcal{PM}_{\mathrm{Lip}}(X)$ which is continuous with respect to the metric D and whose restriction onto every $\mathcal{PM}_{\mathrm{Lip}}(A)$, where A is a subset of X, is linear?

Note that the question which corresponds to the above one for the case of partial Lipschitz fuctions is also open; see [13] for the results on simultaneous extensions of partial continuous functions.

Question 5.4. The space $\mathcal{PM}_{\mathrm{Lip}}(X)$ can be endowed also with the topologies of the uniform and pointwise convergence. Are there linear continuous extensions operators from $\mathcal{PM}_{\mathrm{Lip}}(A)$ to $\mathcal{PM}_{\mathrm{Lip}}(X)$, where A is a subset of X, that are continuous in these topologies?

By $\mathcal{UM}_{\mathrm{Lip}}(Y)$ we denote the set of all Lipschitz ultrametrics on a subset Y of a zero-dimensional metric space. In [14] (see also [15]) it is proved that there exists a continuous extension operator that extends ultrametrics defined on a closed subspace of a zero-dimensional compact metric space and preserves the operation max.

Question 5.5. Given a subset A of a zero-dimensional metric space X, is there a continuous extension operator for Lipschitz ultrapseudometrics, $u\colon \mathcal{UM}_{\mathrm{Lip}}(A) \to \mathcal{UM}_{\mathrm{Lip}}(X)$, that preserves the operation max? is homogeneous?

Apparently, the arguments of the proof of Theorem 4.1 can be applied to another situations in which there is no continuous linear extension operator for a given class of functions; see, e.g., [16]. We are going to return to these questions elsewhere.

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